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Self-avoiding walks in wedges

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Abstract. We consider the number of self-avoiding walks confined to a subset $\mathbb{Z}^d(f)$ of the d-dimensional hypercubic lattice $\mathbb{Z}^d$, such that the coordinates $(x_1, x_2, \ldots, x_d)$ of each vertex in the walk satisfy $x_1 \geq 0$ and $0 \leq x_k \leq f_k(x_1)$ for $k = 2, 3, \ldots, d$. We show that if $f_k(x) \to \infty$ as $x \to \infty$, the connective constant of walks in $\mathbb{Z}^d(f)$ is identical to the connective constant of walks in $\mathbb{Z}^d$. We also explore conditions on f_k which lead to a smaller connective constant for walks in $\mathbb{Z}^d(f)$ and, in particular, consider walks between two parallel $(d-1)$-dimensional hyperplanes. Finally we contrast some of these results with recent work by Grimmett on percolation on subsets of the square lattice.

1. Introduction

Self-avoiding walks with various geometrical constraints have been studied as models of polymer adsorption (Hammersley *et al* 1982), steric stabilisation of dispersions (Dolan and Edwards 1975, Levine *et al* 1978) and the behaviour of polymers in slits and tubes (Daoud and de Gennes 1977, Wall *et al* 1977, Wall and Klein 1979). Because of the analogy between the polymer problem and the behaviour of magnetic systems (Daoud *et al* 1975) those models are also useful in understanding surface magnetism and the behaviour of slabs of magnetic material (Binder 1983).

Cardy (1983) has recently considered the critical behaviour of a magnet in d dimensions, bounded by two $(d-1)$-dimensional hyperplanes which meet at an edge. In particular he used renormalisation group methods to study the critical exponents for correlation functions and susceptibilities for spins close to the edge. The polymer analogue of this problem can be obtained by considering D-dimensional spins on each d-dimensional lattice site and letting $D \to 0$, (Daoud *et al* 1975, Barber *et al* 1978). One is then interested in the numbers of self-avoiding walks starting at the origin and confined to lie between these planes, and the subsets of these walks which either return to one of these planes, or to the edge, at their last step.

In this paper we shall prove that these three classes of self-avoiding walks have the same connective constant (κ) as self-avoiding walks with no such geometrical restrictions. In fact we study the more general problem of walks on a d-dimensional hypercubic lattice such that the coordinates $(x_1, x_2, \ldots, x_d)$ of each vertex in the walk satisfy $0 \leq x_1$, $0 \leq x_k \leq f_k(x_1)$ for $2 \leq k \leq d$. We establish sufficient conditions on the f_k to ensure that the walks, subject to these restrictions, have connective constant κ. We also discuss conditions on f_k which are sufficient to ensure that the connective constant is strictly less than κ and, in particular, study walks confined between two parallel

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planes, a distance L apart. In this case we show that the connective constant $\kappa(L)$ is a strictly monotone function of L and tends to κ as L tends to infinity.

Finally, we compare some of our results with recent work by Grimmett (1981, 1983) on percolation on subsets of the square lattice. The behaviour for percolation is characteristically different to that for self-avoiding walks.

2. Definitions and notation

We work in d-dimensional Euclidean space $\mathbb{R}^d$ with $d \geq 2$, writing $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ for a typical point and its coordinates: here bold-face type signifies vectors, so that a typical sequence of points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ should not be confused with the coordinates $x_1, x_2, \ldots$ of $\boldsymbol{x}$. The vector inequality

$$\boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b} \tag{2.1}$$

is an abbreviation for

$$a_k \leq x_k \leq b_k \qquad (k = 1, 2, \ldots, d) \tag{2.2}$$

and the set of all $\boldsymbol{x}$ lying in a closed interval of the form (2.1) is called a *box*. A box has $2d$ *faces*, these being the $(d-1)$-dimensional subsets obtained by substituting equality for one of the $2d$ inequalities in (2.2); but not all of these faces will be distinct unless $\boldsymbol{a} < \boldsymbol{b}$. The smallest box that contains some bounded set of points S is called the *snug* box for S. A *translate* of S is the image of S under a transformation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$ for some constant vector $\boldsymbol{a}$. We write $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ for the unit vectors in the positive directions of the coordinate axes $\mathbb{R}^d$.

Given a set of non-negative functions $f_2, f_3, \ldots, f_d$, the set of all points $\boldsymbol{x}$ whose coordinates satisfy

$$x_1 \geq 0, \qquad f_k(x_1) \geq x_k \geq 0 \qquad (2 \leq k \leq d) \tag{2.3}$$

is called an *f-wedge* and denoted by $\mathbb{R}^d(f)$. The hypercubic lattice $\mathbb{Z}^d$ is the set of all points of $\mathbb{R}^d$ with integer coordinates; and $\mathbb{Z}^d(f)$ denotes the intersection of $\mathbb{Z}^d$ and $\mathbb{R}^d(f)$.

An *n-stepped walk* is an ordered sequence $z_0, z_1, \ldots, z_n$ of points of $\mathbb{Z}^d$ such that successive points are unit distance apart; and it is an n-stepped *self-avoiding walk* (abbreviated hereafter as an n-SAW) if these $n+1$ points are all distinct. We call the points $z_0, z_1, \ldots, z_n$ the vertices of the walk, we say that a walk w *visits* a point z of $\mathbb{Z}^d$ if z is one of its vertices, and we use adjectives such as *first, earliest, last*, etc with natural reference to the order of the sequence $z_0, \ldots, z_n$. Two walks are *equivalent* if one is a translate of the other; and often we are only interested in the properties of n-SAWs to within equivalence. In that case we can standardise an n-SAW w by taking its first vertex at the origin ($z_0 = \boldsymbol{0}$) and representing it by a sequence of steps $w = \sigma_1 \sigma_2 \ldots \sigma_n$, where each step $\sigma_i = z_i - z_{i-1}$ takes one of the $2d$ possible values $\pm \boldsymbol{e}_k$ $(k = 1, 2, \ldots, d)$. We write c_n for the number of n-SAWs with $z_0 = \boldsymbol{0}$ and $C(n)$ for the set of these walks; and we know (Hammersley 1957) that

$$\lim_{n \to \infty} n^{-1} \log c_n = \kappa, \tag{2.4}$$

where κ, which depends only on d, is the so-called *connective constant* of $\mathbb{Z}^d$. (Note, however, that the term *connective constant* is sometimes used instead for $\mu = \mathrm{e}^{\kappa}$; so care is needed to identify the usage in any particular paper in the literature.) Similarly we write $c_n(f)$ for the number of n-SAWs with $z_0 = \mathbf{0}$ with all their vertices in $\mathbb{Z}^d(f)$. We are then interested in conditions upon f such that

$$\lim_{n\to\infty} n^{-1} \log c_n(f) = \kappa \tag{2.5}$$

may be true or false. We shall see, in due course, that a sufficient condition for the truth of (2.5) is

$$\lim_{x\to\infty} f_k(x) = \infty \qquad (2 \leq k \leq d), \tag{2.6}$$

and conversely that (2.5) is false if f_k is a bounded function of x for at least one value of k.

An n-stepped *self-avoiding circuit* (abbreviated as n-SAC) is an $(n-1)$-SAW whose first and last vertices z_0 and z_{n-1} are unit distance apart. If $z_0, z_1, \ldots, z_{n-1}$ are the successive vertices of an n-SAC, then any cyclic permutation of these vertices is also an n-SAC; and so too is the reverse permutation $z_{n-1}, z_{n-2}, \ldots, z_0$ and all the cyclic permutations of this reverse permutation. The resulting set of $2n$ n-SACs that originate from a given n-SAC may be regarded as a single geometrical entity called an *n-stepped self-avoiding polygon* (abbreviated as n-SAP). (An n-SAC is a rooted, directed n-SAP.) Two n-SAPs are equivalent if one is a translate of the other. Thus a standardised representative of an equivalence class of n-SAPs is an n-SAP that visits the origin of $\mathbb{Z}^d$, because each n-SAP is an unrooted undirected cycle. We write p_n for the number of inequivalent n-SAPs visiting the origin, and $p_n(f)$ for the number of those having all their vertices in $\mathbb{Z}^d(f)$. Parity considerations on $\mathbb{Z}^d$ require that any n-SAP must have n even; and hereafter we tacitly adopt the convention that n is always even in any statement involving n-SAPs. We know (Hammersley 1961) that

$$\lim_{n\to\infty} n^{-1} \log p_n = \kappa, \tag{2.7}$$

and so we enquire whether or not

$$\lim_{n\to\infty} n^{-1} \log p_n(f) = \kappa. \tag{2.8}$$

If there is some integer $x_0 \geq 0$ such that $f_k(x_0) = 0$ for all $k = 2, 3, \ldots, d$, there can be no n-SAPs in $\mathbb{Z}^d(f)$ for arbitrarily large n because they could not enjoy two different routes forwards and backwards through the resulting bottleneck in the hyperplane $x_1 = x_0$. In discussing (2.8) we therefore impose, without essential loss of generality, the condition

$$f_d(x) \geq 1. \tag{2.9}$$

We shall prove that (2.6) and (2.9) are sufficient conditions for the truth of

$$\liminf_{n\to\infty} n^{-1} \log p_n(f) \geq \kappa, \tag{2.10}$$

and hence for the truth of (2.5) and (2.8) in view of the trivial inequality

$$p_n(f) \leq c_{n-1}(f) \leq c_{n-1} \tag{2.11}$$

coupled with (2.4). (Note that (2.9) is no longer needed for (2.5), because a separate return path to the origin is then unnecessary.)

3. Bellman's theorem

A *pattern P* is a prescribed finite sequence of steps, and is said to *occur* on a walk w if it is a subwalk of w. Successive occurrences may overlap: for example, when $d=2$, we may write $w=\mathrm{NE^3SE^2N}$ for the walk whose successive steps are north, east, east, east, south, east, east, north and the pattern $\mathrm{P}=\mathrm{E}^2$ occurs three times on w. Given any particular pattern P, we can always find a Pólya walk on which P occurs as often as we wish; but the situation is quite different if w is a SAW. Obviously P cannot occur on any SAW unless P is a SAW itself. It is also possible to find patterns P that may occur once or twice on SAWs in $\mathbb{Z}^d$ though never more often than twice. Thus the 12-stepped pattern $\mathrm{P}=\mathrm{NWS^2E^4N^2WS}$ occurs twice on the 35-SAW $w=\mathrm{PWN^2E^5S^2WP}$, but P can never occur thrice on any SAW in $\mathbb{Z}^2$; and this sort of 'lobster-pot' or 'trapdoor' construction can readily be generalised to $\mathbb{Z}^d$ for any $d\geqslant 2$. On the other hand, if $k>2$, there are no patterns that can occur k times on a SAW but never $k+1$ times, for the following reason.

Theorem. If a particular pattern P occurs three times on some SAW in $\mathbb{Z}^d$, then P can occur infinitely often on an infinite SAW in $\mathbb{Z}^d$.

This theorem, intuitively motivated by the fact that a walk has just two ends, needs careful proof because successive occurrences of P may overlap and become highly entangled with each other in the rich topology of high-dimensional space.

Let P_1, P_2, P_3 be the first, second and third occurrences of P on a SAW w; and write B_1, B_2, B_3 for the respective snug boxes of P_1, P_2, P_3. These three boxes all have the same shape, size, and orientation, because the same pattern P is snug in each; but B_1, B_2, B_3 must occupy different positions in $\mathbb{R}^d$, for $\mathrm{B}_1=\mathrm{B}_2$ would imply that P_1 and P_2 exactly coincided in violation of the self-avoidance of w. Hence at least one complete face of B_1 must lie strictly outside B_2. But P_1, being snug in B_1, visits every face of B_1; so P_1 contains a vertex outside B_2. This vertex cannot lie in any overlap of P_1 and P_2, because P_2 lies wholly in B_2; so this vertex precedes all vertices of P_2 on w. Hence there is a non-empty class of vertices on w lying outside B_2 and preceding all vertices of P_2; and this class possesses a latest vertex V_1 on w. Similar consideration of $\mathrm{B}_2\neq\mathrm{B}_3$ guarantees the existence of an earliest vertex V_3 outside B_2 and succeeding all vertices of P_2. Let w_2 denote the subwalk of w from V_1 to V_3. Discard all steps of w except the steps of w_2. Then w_2 has just two vertices outside B_2, and there exists a box B_4 (whose *interior* strictly contains B_2) such that only the first and last vertices of W_2 lie on the faces of B_4. (B_4 is *not* snug for w_2.) If V_1 and V_3 do not lie on opposite faces of B_4, we may append some extra steps to the end of w_2 to make this happen; and we can ensure that the extended walk is still a SAW by suitably utilising steps confined to the surface of B_4. Finally we can add one more step to the end of this extended w_2 to reach a vertex outside B_4. This yields a pattern Q, which is a SAW containing the pattern P, such that the infinite repetition QQQ ... is an infinite SAW on which P occurs infinitely often.

This theorem is known as Bellman's theorem on account of the Bellman's remark 'What I tell you three times is true' (Carroll 1876); but Carroll needed 42 boxes 'all carefully packed' in his account.

4. Proof of (2.10)

Prescribe an even integer $N > 0$ and consider the set of points of $\mathbb{Z}^d$ satisfying

$$x_1 = 0, \qquad 0 \leqslant x_k \leqslant N \qquad (2 \leqslant k \leqslant d). \tag{4.1}$$

This set contains $M = (d-1)N^{d-1}$ unordered pairs of adjacent points, which we index in some fixed manner by $i = 1, 2, \ldots, M$. Now consider an N-SAP translated so that its snug box has the form

$$\mathbf{0} \leqslant \boldsymbol{x} \leqslant \boldsymbol{b}. \tag{4.2}$$

No coordinate of $\boldsymbol{b}$ can exceed N, and the N-SAP must pass through at least one pair of adjacent vertices on the face $x_1 = 0$. So we can index this N-SAP by the index i ($1 \leqslant i \leqslant M$), using the smallest available index if the N-SAP passes through more than one pair of adjacent vertices on the face $x_1 = 0$. Similarly, we can classify the N-SAP by a second index j ($1 \leqslant j \leqslant M$), corresponding to a pair of adjacent vertices that it visits on the face $x_1 = b_1$ of its snug box (using, of course, the *same* fixed indexing as in (4.1) but with $x_1 = b_1$). The indexing i, j partitions the set of all inequivalent N-SAPS into M^2 disjoint subsets P_{ij}, of which some subsets may be empty.

Now consider two N-SAPS, the first a member of P_{ij} fitted into a snug box (4.2), and the second a member of P_{jk} translated to fit into a snug box

$$b_1 + 1 \leqslant x_1 \leqslant b_1', \qquad 0 \leqslant x_k \leqslant b_k' \qquad (2 \leqslant k \leqslant d). \tag{4.3}$$

Since the second suffix of P_{ij} equals the first suffix of P_{jk}, the two SAPS will have parallel adjacent edges on the faces $x_1 = b_1$ of the first SAP and $x_1 = b_1 + 1$ of the second SAP. We can remove these two edges and replace them by a pair of edges between the two faces, thus concatenating the two SAPS and constructing a $2N$-SAP. Further, given any $2N$-SAP constructed in this manner, we can reverse the process and uniquely recover the pair of N-SAPS from which it was derived. Hence, if p_{ij} denotes the number of N-SAPS in the subset P_{ij}, and $\boldsymbol{P}$ is the $M \times M$ matrix $\boldsymbol{P} = (p_{ij})$, the total number of $2N$-SAPS constructed by this concatenation and belonging to the subset P_{jk} is the (j, k)-element in $\boldsymbol{P}^2$. By repeated concatenation, we can construct a set of inequivalent Nr-SAPS, whose number is the sum of the elements in the matrix $\boldsymbol{P}^r$. Moreover all these Nr-SAPS can be translated to fit into snug boxes of the form

$$a_1 \leqslant x_1 \leqslant b_1, \qquad 0 \leqslant x_k \leqslant b_k \leqslant N \qquad (2 \leqslant k \leqslant d); \tag{4.4}$$

and all these snug boxes will be contained in $\mathbb{Z}^d(f)$ provided that

$$a_1 \geqslant \alpha(N) \tag{4.5}$$

for some number α depending on N, in view of (2.6).

We now wish to construct a polygon, in $\mathbb{Z}^d(f)$, which visits the origin and contains the edge between the points $(u_1, u_2, \ldots, u_q, \ldots, u_d)$ and $(u_1, u_2, \ldots, u_q + 1, \ldots, u_d)$ in the face $x_1 = u_1 = b_1$ of its snug box $\mathbf{0} \leqslant \boldsymbol{x} \leqslant \boldsymbol{b}$. If we write $\boldsymbol{e}_k^u$ for a succession of u steps in the direction $\boldsymbol{e}_k$ and $\bar{\boldsymbol{e}}_k^u$ for a succession of u steps in the direction $-\boldsymbol{e}_k$, a polygon satisfying these conditions can be written as

$$\Pi_1 = \boldsymbol{e}_1{}^{u_1-1}\boldsymbol{e}_2{}^{u_2} \ldots \boldsymbol{e}_d{}^{u_d}\boldsymbol{e}_1\boldsymbol{e}_q\bar{\boldsymbol{e}}_d\bar{\boldsymbol{e}}_1\bar{\boldsymbol{e}}_q\bar{\boldsymbol{e}}_d{}^{u_d-2}\bar{\boldsymbol{e}}_{d-1}^{u_{d-1}} \ldots \bar{\boldsymbol{e}}_2{}^{u_2}\bar{\boldsymbol{e}}_1{}^{u_1-1}$$

provided that $u_d \geqslant 2$. If $u_d = 0$ or 1, an appropriate polygon is

$$\Pi_2 = \boldsymbol{e}_1{}^{u_1-1}\boldsymbol{e}_2{}^{u_2} \ldots \boldsymbol{e}_d{}^{u_d}\boldsymbol{e}_1\boldsymbol{e}_q\boldsymbol{e}_d\bar{\boldsymbol{e}}_1\bar{\boldsymbol{e}}_q\bar{\boldsymbol{e}}_d{}^{u_d}\bar{\boldsymbol{e}}_{d-1}^{u_{d-1}} \ldots \bar{\boldsymbol{e}}_2{}^{u_2}\bar{\boldsymbol{e}}_1{}^{u_1-1}.$$

After translation, Π_1 fits into the snug box

$$0 \leqslant x_k \leqslant u_k + \delta_k \tag{4.6}$$

with

$$\delta_k = \begin{cases} 1 & \text{if } k = q \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

and Π_2 fits into the snug box (4.6) with

$$\delta_k = \begin{cases} 1 & \text{if } k = q \text{ or } d \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

These SAPs will belong to some class P_{ij} provided that $u_k < N$ for $2 \leqslant k \leqslant d$. By choosing an integer q $(2 \leqslant q \leqslant d)$ and integers $u_k < N$, $k = 2, \ldots, d$, we can construct for some i_0 a SAP belonging to $P_{i_0 j}$ for any given value of $j = 1, 2, \ldots, M$, and this polygon can be concatenated in front of any Nr-SAP by taking $a_1 = u_1 + 1$. The SAP formed by this concatenation will be contained in $\mathbb{Z}^d(f)$ provided that

$$u_1 \geqslant \beta(N) \tag{4.9}$$

for some integer $\beta(N)$, because of (2.6). By choosing u_1 appropriately, subject to (4.9) the SAP can be constructed with n edges where n is any sufficiently large even integer. This n will have the form

$$n = Nr + A \tag{4.10}$$

where A is the number of edges in Π_1 or Π_2, and r is any positive integer. Hence

$$p_n(f) \geqslant \text{sum of elements of } \boldsymbol{P}^r \geqslant p_{jk}p_{kj}p_{jk} \ldots \quad (r \text{ terms}) \tag{4.11}$$

for any pair of indices j, k. By reflecting any N-SAP in $x_1 = 0$ it is easy to see that $\boldsymbol{P}$ is a symmetric matrix. If we choose j, k so that p_{jk} is the largest element in $\boldsymbol{P}$ we have

$$p_n(f) \geqslant (p_N/M^2)^r \tag{4.12}$$

because p_N is the sum of all elements in $\boldsymbol{P}$. Hence

$$n^{-1} \log p_n(f) \geqslant N^{-1}(1 - A/n) \log p_N - 2N^{-1}(1 - A/n) \log[(d-1)N^{d-1}]. \tag{4.13}$$

Letting $n \to \infty$ in (4.13) we have

$$\liminf_{n\to\infty} n^{-1} \log p_n(f) \geqslant N^{-1}\{\log p_N - 2\log[(d-1)N^{d-1}]\}. \tag{4.14}$$

Since the LHS of (4.14) is independent of N, we can now let $N \to \infty$ and (2.10) follows from (4.14) and (2.8). The influence of f upon the rate of approach to the limit can be estimated from (4.13).

5. Constrained self-avoiding walks and patterns

In § 4 we discussed a sufficient condition on the functions f_k defining an f-wedge which ensured that the connective constant of SAWs in an f-wedge would be identical to that for SAWs on the complete lattice. We now turn to the opposite case and look for sufficient conditions for there to be exponentially fewer SAWs in the f-wedge than on the complete lattice. The idea is to make use of a theorem of Kesten (1963) on patterns.

If P is any (finite) pattern and $c_n(\varepsilon, \mathrm{P})$ is the number of n-step self-avoiding walks in which the pattern P occurs *at most* εn times then there exists a value of $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} [c_n(\varepsilon, \mathrm{P})/c_n]^{1/n} < 1 \tag{5.1}$$

provided that there exists a self-avoiding walk in which P appears more than twice.

The corollary which we require is as follows: if there exists a pattern P which occurs more than twice in at least one self-avoiding walk in $\mathbb{Z}^d$ but does not occur in any self-avoiding walk in $\mathbb{Z}^d(f)$ then

$$\limsup_{n\to\infty} n^{-1} \log c_n(f) < \kappa. \tag{5.2}$$

Since $c_n(f) \leqslant c_n(0, \mathrm{P})$ (5.2) follows immediately from (5.1).

We now consider two examples.

Example 1. Consider a SAW on $\mathbb{Z}^d$ confined to lie in or between the hyperplanes $x_d = 0$ and $x_d = L$. Let $c_n(L)$ be the number of these walks which start at the origin. If N and S represent steps in the positive x_d and negative x_d directions, and E and W represent steps in the positive x_1 and negative x_1 directions, then the pattern

$$\mathrm{P} = \mathrm{EN}^{L+1}\mathrm{ES}^{L+1}$$

does not occur in any SAW confined between $x_d = 0$ and $x_d = L$. However, P does occur more than twice in a SAW in $\mathbb{Z}^d$ (the walk P^3 for example) so that

$$\limsup_{n\to\infty} n^{-1} \log c_n(L) < \kappa \tag{5.3}$$

for any finite L.

Example 2. Consider a SAW on $\mathbb{Z}^2(f)$ with $f_2(x) = \mathrm{e}^x \sin^2(\pi x/L)$. $f_2(x)$ is unbounded above but is zero when x is an integral multiple of L. The pattern

$$\mathrm{P} = \mathrm{E}^{L+1}\mathrm{NE}^{L+1}$$

occurs more than twice in a SAW in $\mathbb{Z}^2$(e.g. in the walk P^3) but does not occur in any SAW in $\mathbb{Z}^2(f)$. Hence

$$\limsup_{n\to\infty} n^{-1} \log c_n(f) < \kappa. \tag{5.4}$$

A similar argument can easily be constructed for any function f_2 which *periodically* takes *any prescribed* finite value. However, if f_2 only returns to the prescribed finite value aperiodically and sufficiently increasingly rarely, then (5.4) could be false.

6. Walks between two parallel planes

In this section we consider the subset $C(n, L) \subseteq C(n)$ such that the walks lie in or between the two hyperplanes $x_d = 0$ and $x_d = L$. If we write $x_k(l)$ for the kth coordinate of the lth vertex in the walk, a walk is a member of $C(n, L)$ if

$$0 \leqslant x_d(l) \leqslant L, \qquad l = 0, 1, \ldots, n. \tag{6.1}$$

It will be convenient to construct a subset $B(n, L) \subseteq C(n, L)$ such that

$$x_1(0) < x_1(l) \leqslant x_1(n) \qquad l = 1, 2, \ldots, n$$

and

$$x_d(n) = 0. \tag{6.2}$$

If the numbers of members of $B(n, L)$ and $C(N, L)$ are $b_n(L)$ and $c_n(L)$ respectively it is straightforward (Whittington 1983) to show that

$$0 < \sup_{n>0} n^{-1} \log b_n(L) = \lim_{n\to\infty} n^{-1} \log b_n(L) = \lim_{n\to\infty} n^{-1} \log c_n(L) \equiv \kappa(L) < \infty. \tag{6.3}$$

The arguments of § 5 establish that

$$\kappa(L) < \kappa \tag{6.4}$$

for any finite L. Here we shall prove that $\kappa(L)$ is a strictly monotone increasing function of L and that

$$\lim_{L\to\infty} \kappa(L) = \kappa. \tag{6.5}$$

We define the subset $P(n, L) \subseteq B(n, L)$ such that a walk in $P(n, L)$ cannot be decomposed into two walks, one from $B(m, L)$ and one from $B(n-m, L)$, for any m. For this reason we call $P(n, L)$ the set of *prime walks*. If $p_n(L)$ is the number of members of $P(n, L)$ we can write

$$b_n(L) = \sum_{m=1}^{n} p_m(L) b_{n-m}(L); \tag{6.6}$$

and if we form generating functions

$$\mathscr{P}(x, L) = \sum_m p_m(L) x^m \tag{6.7}$$

and

$$\mathscr{B}(x, L) = \sum_m b_m(L) x^m, \tag{6.8}$$

we obtain

$$\mathscr{B}(x, L) = \mathscr{P}(x, L)/[1 - \mathscr{P}(x, L)]. \tag{6.9}$$

We now remark that $\mathscr{B}(x, L) \to \infty$ as $x \to \exp[-\kappa(L)]$ from below. Indeed this follows by an argument exactly analogous to an argument of Kesten (1963, theorem 5), and we shall not repeat the details here. This, together with (6.9), implies that

$$\mathscr{P}(\mathrm{e}^{-\kappa(L)}, L) = 1. \tag{6.10}$$

But if $\kappa(L) = \kappa(L')$, $L > L'$, then $\mathscr{P}(x, L) = 1$ and $\mathscr{P}(x, L') = 1$ must have a common root at $x = \mathrm{e}^{-\kappa(L)}$. This is impossible since $p_m(L)$ is non-negative and $p_m(L) \geqslant p_m(L')$ with strict inequality for some values of m. Hence

$$\kappa(L) > \kappa(L'), \qquad L > L'. \tag{6.11}$$

If we put $f_d(x)=L$ for $x\geqslant 0$, (4.14) becomes

$$\begin{aligned}\kappa &\geqslant \liminf_{n\to\infty} n^{-1}\log c_n(L)\\ &\geqslant \liminf_{n\to\infty} n^{-1}\log p_n(L)\\ &\geqslant L^{-1}\{\log p_L - 2\log[(d-1)L^{d-1}]\}\end{aligned} \tag{6.12}$$

The RHS of (6.12) tends to κ as $L\to\infty$. Hence

$$\lim_{L\to\infty} \kappa(L)=\kappa. \tag{6.13}$$

7. Discussion

We have been concerned with the number of self-avoiding walks confined to lie in a region between the x axis and a curve $y=f(x)$, and the generalisation of this restriction to d dimensions. Provided that $f\to\infty$ as $x\to\infty$ we have shown that the connective constant for such walks is equal to the connective constant for the complete lattice. In particular, this establishes that the three kinds of walks in a wedge, corresponding to the zero spin space dimension limit of Cardy's problem (Cardy 1983) have the same connective constant as the complete lattice, for all positive wedge angles.

In § 5 we discussed the use of a theorem by Kesten, on patterns in self-avoiding walks, to establish conditions on f sufficient to ensure that only an exponentially small fraction of self-avoiding walks survive the condition that they are restricted to an f-wedge, and in § 6 we discussed the particular case of walks confined between parallel lines or planes, in more detail.

The result derived in § 4 should be contrasted with a result of Grimmett (1983) for the bond percolation problem. Grimmett investigated the percolation probability on the subset of the square lattice $\{(x,y)\in\mathbb{Z}^2\colon 0\leqslant y\leqslant f(x), 0\leqslant x\}$. If $f(x)=a\log(x+1)$ he showed that the critical probability for percolation $p_c(a)$ is a continuous strictly decreasing function of a with $p_c(a)\to 1$ as $a\to 0+$ and $p_c(a)\to\frac{1}{2}$, as $a\to\infty$. Hence, if $f(x)/\log x\to\infty$ as $x\to\infty$, $p_c=\frac{1}{2}$ and if $f(x)=o(\log x)$, $p_c=1$. By contrast, in the self-avoiding walk problem the connective constant is independent of f if $f\to\infty$ *no matter how slowly.*

Grimmett's arguments depend strongly on the lattice being two-dimensional, but weaker results can be obtained for $d>2$. We consider $\mathbb{Z}^d(f)$ and let $g(x)$ be the number of bonds in the x_1 direction from $x_1=x$ to $x_1=x+1$ in $\mathbb{Z}^d(f)$. Let A_x be the event that all of these bonds are blocked, and q be the probability that a given bond is blocked. Since the events $\{A_x\}$ are independent it follows from the Borel-Cantelli lemma that infinitely many A_x will occur with probability one if

$$\sum_x q^{g(x)}=\infty. \tag{7.1}$$

If, for $x\geqslant x_0$, $g(x)\leqslant a\log x$ then

$$\sum_{x\geqslant x_0} q^{g(x)}\geqslant \sum_{x\geqslant x_0} x^{a\log q} \tag{7.2}$$

which is infinite provided that $a\log q\geqslant -1$. In this case no percolation occurs and hence we have shown that

$$p_c(f) \geqslant 1 - e^{-1/a}. \tag{7.3}$$

In particular, if $g(x) = o(\log x)$ then $p_c = 1$.

We can also show that $p_c < 1$ if $f_k(x) \geqslant \varepsilon(\log x)^{1/d}$ for $k \geqslant 2$, for $x \geqslant x_0$, for any positive ε. We sketch the argument only for $d = 3$. If we consider a plane $x_1 = x$, we can order the $g(x)$ vertices in this plane which are also in $\mathbb{Z}^3(f)$ as $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(1, 2)$, $(0, 2)$, $(0, 3)$, $(1, 3)$, ... where we list only the (x_2, x_3) coordinates. We now consider the sublattice consisting of these vertices for all x_1 planes, the bonds between adjacent x_1 planes and bonds within planes which join adjacent members in the foregoing ordering. By the containment theorem (Fisher 1961) this lattice percolates *less* readily than $\mathbb{Z}^3(f)$. However, this lattice can be deformed into a subset of the two-dimensional square lattice ('unroll' the vertices in the ordered set above) so we can make use of Grimmett's result (that $p_c(a)$ is a strictly decreasing function of a) to establish that $p_c < 1$ provided that $f_k(x) \geqslant \varepsilon(\log x)^{1/2}$, $\varepsilon > 0$, $x \geqslant x_0$, $k = 2, 3$. Likewise

$$f_k(x)/(\log x)^{1/2} \to \infty \qquad \text{as } x \to \infty \qquad (k = 2, 3) \tag{7.4}$$

implies $p_c(f) \leqslant \frac{1}{2}$. Under additional conditions, it might be true that (7.4) implies that $p_c(f) = p_c$(i.e. the critical probability of $\mathbb{Z}^3$). Grimmett (1984) has verified our conjecture that (7.4) could be replaced by

$$g(x)/\log x \to \infty \qquad \text{as } x \to \infty. \tag{7.5}$$

We can now ask for corresponding results for the Ising problem. If we consider the Ising problem on an infinite subset of the square lattice, how does the critical temperature depend on the subset chosen? There are a few results on the 'slab' geometry discussed in § 6. If $d = 2$ there is no phase transition, on any finite width slab, but if $d = 3$ there is series analysis work on the dependence of the critical temperature (T_c) on the width (L) (Allan 1970), though we are not aware of any rigorous results on this question. However, the spherical model has been solved in a slab geometry and the dependence of T_c on L is known for this problem (see e.g. Watson 1972).

For the Ising problem on an f-wedge, even in two dimensions, almost nothing seems to be known. It is tempting to conjecture that the behaviour follows the percolation pattern rather than the self-avoiding walk pattern and it would be interesting to investigate this. The only result of which we are aware is that the half-plane and the plane square lattices have the same critical temperature (McCoy and Wu 1973). On the other hand, the critical exponents of the correlation function and of the spontaneous magnetisation do depend upon the wedge angle (Barber *et al* 1984).

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